Transmission-Zero Bounds for Large Space Structures, with Applications

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Many large space structure control problems lead quite naturally to the application of an optimal regulator, so the transmission zeros of the open-loop system give fundamental information about the speed of response achievable by the closed-loop system. Despite the importance of this and other well-known zeros properties, little attention has been given to the transmission zeros of large space structures, except for the special case of a rigid spacecraft with flexible appendages. The object of this paper is to remedy this deficiency. In particular, it is proved that the zeros of a structure with colocated sensors and actuators must lie in a region of the complex plane that is defined by its natural frequencies and damping ratios. This generic result, a consequence of the special form of the equations of motion of structural dynamics, admits a very simple graphical interpretation: it is the generalization of the classical pole-zero interlacing property of undamped single-input/single-output structures. The number of sensor/actuator pairs, and their locations, specify where in the permissible region transmission zeros actually lie, thus quantifying the effect of sensor/actuator placement on closed-loop system performance. These points are illustrated by simple examples.

Introduction

MANY problems in the control of large space structures (LSS) can be expressed quite naturally in an optimization form.[1,2] For example, the objective when controlling the shape of a flexible antenna is generally to minimize its root-mean-square (rms) surface deflection, while that in many LSS maneuvers is to keep the structural vibrations set up as small as possible without applying excessive control inputs. Consequently, the *linear optimal regulator*[3] has proved to be of great relevance to LSS control.

Optimal regulators have been the subject of extensive control theory research over the past two decades, and many of their properties have been well established. Of particular interest here is the fact[3,4] that the *transmission zeros* of the open-loop system have been shown to give fundamental information about the speed of response achievable by the optimal closed-loop system; they also have many other well-documented applications in linear systems theory, for instance, to decoupling problems.[4] Thus, a study of these zeros (which depend explicitly on the positions at which sensors are located) is a necessary part of any full analysis of the dynamics of a given structure. Despite this importance, little attention has been given to the transmission zeros of large space structures, except in the special case of a rigid spacecraft with flexible appendages.[5,6] This is perhaps due to the fact that any typical LSS is, of necessity, a multi-input/multi-output system, so the definition and physical significance of its zeros are not as intuitively clear as for a single-input/single-output system. In particular, these zeros are not related to those of the scalar transfer functions between individual inputs and outputs.

Thus, the object of this paper is, first, to clarify the role of the transmission zeros of a given LSS, and, second, to prove that, if all sensors and actuators are colocated, these zeros must lie in a region of the complex plane that is defined by the natural frequencies and damping ratios of the structure. This generic result, a consequence of the special form of the equations of motion of structural dynamics, admits a very simple graphical interpretation: it will be shown to be a generalization of the classical pole-zero interlacing property of undamped single-input/single-output structures.[5] The number of sensor/actuator pairs, and their locations, specify where, in this permissible region, transmission zeros actually lie (they are given as the eigenvalues of a *constrained modes* problem[7]). Thus, the close relationship between transmission zeros and optimal regulator performance allows the effects of sensor/actuator placement on the resulting closed-loop dynamics to be readily quantified. These points will be illustrated by simple examples.

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Poles and Transmission Zeros

Consider an n-mode model for the structural dynamics of a nongyroscopic, noncirculatory LSS with m actuators and p sensors, not necessarily colocated. This can be written as

$$M\ddot{q} + C\dot{q} + Kq = Vu, \quad y = W_r\dot{q} + W_dq \tag{1}$$

where q is the vector of generalized coordinates, u that of applied actuator inputs, and y that of sensor outputs. The mass, stiffness, and damping matrices of the structure satisfy $M = M^T > 0$, $K = K^T \geq 0$, and $C = C^T \geq 0$, respectively, while the control influence matrix V is of full column rank.

Taking the Laplace transform of Eq. (1) yields the frequency-domain *polynomial matrix representation*[8]

$$P(s)q(s) = Vu(s), \quad y(s) = W(s)q(s) \tag{2}$$

for the given LSS, where $P(s) = s^2M + sC + K$ and $W(s) = sW_r + W_d$. Note that $P(s)$ is symmetric, i.e., Eq. (2) respects the special structure of the LSS equations of motion. This is in contrast to the standard *state-space representation* $\dot{x} = Ax + Bu$, $y = Cx$ with $x = (\dot{q}^T, q^T)^T$, where A no longer preserves this useful symmetric structure.

The *poles* (or resonances) of the given system are those complex $s_i = \sigma + j\omega$ at which it is possible to obtain a nonzero output evolving with time as $\exp(s_it)$, i.e., as $e^{\sigma t}\cos\omega t$, in response to an identically zero input. This occurs when the initial condition q_i on $q(t)$ is nonzero and chosen to satisfy $P(s_i)q_i = 0$, so clearly $P(s_i)$ must be of less than full rank for any pole. Conversely, the *transmission zeros*[4] are those complex s_i at which it is possible to apply a nonzero input and get an identically zero output in response, again for suitable initial conditions. Defining the system *transfer matrix*[8] $T(s)$ as the rational matrix satisfying $y(s) = T(s)u(s)$, the transmission zeros are clearly those s_i for which $T(s_i)$ is of less than full rank, and the desired output-nulling control is of the form $\exp(s_i t)u_i$, where $T(s_i)u_i = 0$. This is a natural generalization of the single-input/single-output case, where a transmission zero s_i makes the *transfer function* $t(s_i)$ zero, giving $y(s_i) = t(s_i)u_i = 0$ regardless of u_i. Note, however, that the transmission zeros of a multi-input/multi-output system are *not* in general related to the frequencies that make zero any particular scalar transfer function $t_{ij}(s)$ between input j and output i; nor are they related to the transmission zeros of any block of $T(s)$, a problem studied in Ref. 6.

As long as its actuators and sensors have been positioned in such a way as to make it completely *controllable* and *observable*, i.e., so that each mode can be both excited and sensed, an equivalent definition for the transmission zeros of this system is[4] those s_i for which the rank of the *system matrix*

$$S(s) = \begin{pmatrix} P(s) & V \\ -W(s) & 0 \end{pmatrix} \tag{3}$$

is reduced. This polynomial matrix condition is often more convenient to deal with than the original definition involving the rational $T(s)$ [$= W(s)P^{-1}(s)V$, by Eq. (2)]. Furthermore, it helps to clarify the meaning of the *zero modes* of the structure. For, if $S(s_i)$ is of less than full rank, there exists a nonzero u_i and q_i for which

$$\begin{pmatrix} P(s_i) & V \\ -W(s_i) & 0 \end{pmatrix}\begin{pmatrix} q_i \\ -u_i \end{pmatrix} = 0 \tag{4}$$

so applying the nonzero input $\exp(s_i t)u_i$ to the system with the generalized coordinate initial condition q_i gives rise to the identically zero output $y(s_i) = W(s_i)q_i$. The zero mode shape q_i can be regarded as the solution of a *constrained modes problem*,[7] with the constraint being simply that the sensor outputs remain identically zero. As a particularly simple special case, the zero modes of a rigid spacecraft with flexible appendages and a single sensor/actuator pair on the central body are the *appendage-alone modes* described in Ref. 5. Here, q_i corresponds to a natural vibration mode of the appendages in isolation, and the scalar u_i to the torque required to keep the central body at rest in the face of this resonance.

Transmission Zero Applications

The transmission zeros of a linear system possess properties that make them as essential as its poles for a complete description of its dynamic behavior. For instance, they are invariant under *linear state feedback*,[4] a control law that becomes, in LSS terms, a combination of rate and displacement feedback,

$$u = F_r\dot{q} + F_d q \tag{5}$$

Such properties make transmission zeros of great importance in various fundamental control problems. In particular, they specify the fixed poles when inverting or decoupling a square system,[8,9] while in regulation and tracking problems they define the fixed closed-loop poles in the high control gain case. One such design technique that has proved[1,2] to be particularly appropriate for many LSS problems is that based on the *linear optimal regulator*.[3] In this formulation, the aim is to obtain a closed-loop system for which the quadratic cost functional

$$J = \int_0^\infty [y^T(t)Q_0 y(t) + \rho u^T(t)R_0 u(t)]\,\mathrm{d}t \tag{6}$$

is minimized, where $Q_0 = Q_0^T \geq 0$, $R_0 = R_0^T > 0$, and the scalar $\rho \geq 0$. It can be shown[3,4] that the controller that minimizes J can be implemented in the form of a state feedback control law, where the gain matrix is given from the solution of an algebraic Riccati equation.

The closed-loop system produced by an optimal regulator is guaranteed to be stable, i.e., all closed-loop poles are guaranteed to lie somewhere in the left-half plane. Their exact positions, however, are functions of the specific cost index J to be minimized and, in particular, of the relative weightings given to output performance and control effort. Thus, the dynamic behavior of the closed-loop system depends on the values chosen for Q_0, R_0, and ρ: it also depends on the $2n$ poles and $2q$ ($< 2n$) zeros of the open-loop system, in a way made explicit by the *optimal root loci*[3,10,11] as follows. Keeping Q_0 and R_0 fixed, we vary the scalar ρ through its entire allowable range of ∞ to 0 and observe the closed-loop poles that result. Taking the two extreme cases in turn, if $\rho \to \infty$ it is considered far more important to minimize control action than output variations, so the "optimal" closed-loop system will use as little feedback as possible. In fact, if the open-loop system is stable [clearly the case for any damped LSS, as can be seen from Eq. (2)], the poles of the closed-loop system in the limiting case will simply be the open-loop poles: no control effort is wasted in shifting them.

For $\rho \to 0$, on the other hand, the "cheap control" case where no constraint is placed on the allowable controller bandwidth, it is considered worthwhile to use large feedback gains in order to obtain very fast output regulation. In this case, $2q$ closed-loop poles tend to the open-loop transmission zeros (or the mirror images in the imaginary axis of any zeros that are in the right-half plane); the remaining $2(n-q)$ poles tend to infinity at various rates and directions, in patterns known as *Butterworth configurations*. If the open-loop system is *minimum phase*,[4] i.e., if all its transmission zeros are in the left-half plane, there are therefore $2q$ pole zero cancellations in the asymptotic closed-loop system. These make the corresponding modes unobservable in the output $y(t)$ (they are, in fact, zero modes of the closed-loop system), so this response depends only on that of the remaining $2(n-q)$ modes: it is thus extremely fast, as the corresponding asymptotic closed-loop poles are infinite. Note, however, that the unobservable modes dominate the behavior of $q(t)$, and so of the control input $u(t)$, by the feedback control law equation (5). [For a system that is not minimum phase, the finite closed-loop poles also dominate the response of $y(t)$, preventing fast output regulation even for $\rho = 0$: see Ref. 3 for further details.]

The transmission zeros of an LSS also help to give some insight into the effects of using *low-authority control*[12,13] to increase its damping ratios and natural frequencies somewhat before applying an optimal regulator main loop. Assuming the existing m actuator stations are used to implement it, low-authority control makes the resulting poles more stable but leaves the transmission zeros unchanged; this is a consequence of the invariance of transmission zeros under state feedback, together with the observation that such low-authority control is of the state feedback form of Eq. (5). Thus, the cheap control destinations of the optimal root loci are not altered at all by such an implementation of low-authority control—a fact likely to be reflected in the overall closed-loop response. However, this difficulty can be avoided by using a set of low-authority sensor/actuator pairs (e.g., semipassive dampers) that are at different locations from those used for the regulator high-authority control loop. The damping feedback now acts through a different control influence matrix than V in Eqs. (1) and (2), and, therefore, does not equate to state feedback; it can thus alter the transmission zeros of the resulting damped LSS as well as its poles. The low-authority control design problem can be posed as that of choosing damper positions and gains so as to shift all poles and zeros to satisfactory locations in the complex plane.

LSS Transmission Zero Bounds

It is now clear that, if bounds could be placed on the possible values for the transmission zeros of an LSS in terms of its open-loop poles, this would allow its entire optimal root loci to be related to these poles, i.e., to its natural frequencies and damping ratios. A type of relation that would be particularly valuable would be a lower bound on the size of the real parts of all zeros; this would guarantee that none of the cheap control closed-loop modes decay at slower than some given rate. Similarly, lower bounds on the moduli of the zeros would guarantee that no mode has an excessively low natural frequency, while a lower limit on their imaginary parts would do the same for their damped frequencies.

Unfortunately, no such pole-zero relations need hold for a structure with noncolocated sensors and actuators. In fact, the transmission zeros of such an LSS do not even necessarily lie in the left-half plane. Therefore, we shall consider from now on the case of colocated sensors and actuators, where much more satisfying results are possible. As is usual, by "colocated sensors and actuators" we mean that measurements and inputs are made not only at the same physical positions but also along/about the same axes—perhaps *compatible* sensors and actuators would be a better term. Assume initially, without loss of generality, that the system generalized coordinates include the p ($=m$) sensed physical displacements. Clearly, these can be made the first m elements of $\boldsymbol{q}$ by simple reordering, giving in Eq. (1) $V=(B^T,0)^T, W_r=(D_r,0)$, and $W_d=(D_d,0)$, where D_r, D_d, and the nonsingular B are all $(m\times m)$. This implies that $W(s)$ in Eq. (2) is simply $(D(s),0)$, where the $(m\times m)$

$$D(s)=sD_r+D_d \tag{7}$$

so the system matrix of Eq. (3) becomes

$$S(s)=\begin{pmatrix} P_1(s) & P_2(s) & B\\ P_2^T(s) & P_3(s) & 0\\ -D(s) & 0 & 0\end{pmatrix} \tag{8}$$

where $P(s)$ has been partitioned conformally with B and $D(s)$. Entirely analogous results apply in cases where the generalized coordinates do not correspond to physical displacements, for instance in an assumed (or true) modes model, where $\boldsymbol{q}$ consists of modal amplitudes. In such cases, V and $W(s)$ take the form Z^TB and $D(s)Z$ for Z some $(m\times n)$ influence matrix, and $S(s)$ can again be reduced to the structural form of Eq. (8) by means of an orthogonal transformation of $\boldsymbol{q}$ based on the *QR decomposition*[14] of Z^T. See Ref. 15 for further details.

Now, the transmission zeros are those s_i that reduce the rank of $S(s)$; so, by inspection, they are those that reduce the rank of either $D(s)$ (the *sensor zeros*) or $P_3(s)$ (the *structural zeros*). D_r and D_d should clearly be chosen so that the sensor zeros are all in the left-half plane; if, in addition, D_r is nonsingular, it can be shown[11] that the m infinite poles in the asymptotic cheap control case lie on the negative real axis. The resulting closed-loop system then has the highly desirable property of *bounding peaking*,[16] which means that the high speed of response of $\boldsymbol{y}(t)$ does not lead to any initial large transients in control inputs or vibrations unobservable at the sensor stations.

Our primary interest here, however, is in the structural zeros, given from the $(n-m)\times(n-m)$

$$P_3(s)=s^2M_3+sC_3+K_3 \tag{9}$$

Now, a principal submatrix of a positive (semi-) definite matrix is itself positive (semi-) definite,[14] so $M_3=M_3^T>0$, $C_3=C_3^T\geq 0$, and $K_3=K_3^T\geq 0$, and this polynomial matrix is of precisely the same form as the $(n\times n)$ $P(s)=s^2M+sC+K$ that defined the $2n$ stable poles of the structure. Therefore, by analogy, there must be $2q=2q(n-m)$ structural zeros, and they must all lie in the left-half plane. Both of these results are in contrast to the noncolocated case, where no minimum phase guarantee is possible, and the number of zeros depends not only on the number of sensors but also on their positions.[17]

These are actually the only zero relations that hold for a structure with general damping matrix C; however, much more can be proved for various special classes of damping. As motivation, consider the very simple, although not very realistic, case of a single-input/single-output undamped structure. By the above reasoning, this must have $2(n-1)$ zeros, and it is a classical result[5,17] that these alternate with the $2n$ system poles along each half of the imaginary axis. In particular, if the natural frequencies of the structure are $\{\omega_i;\ i=1,\ldots,n\}$, then its poles are $\{\pm j\omega_i\}$ and its zeros $\{\pm jz_i;\ i=1,\ldots,n-1\}$, where

$$\omega_1\leq z_1\leq\omega_2\leq\cdots\leq z_{n-1}\leq\omega_n \tag{10}$$

We now propose to generalize this result to the damped multi-input/multi-output case, showing how the pole locations specify regions in which the zeros must lie. This will be done for the following progressively less restrictive classes of damping:

Undamped system:

$$C=0$$

Proportional damping:

$$C=aM+bK$$

Modal damping:

$$\Phi^TM\Phi=I$$

$$\Phi^TC\Phi=\mathrm{diag}(2\zeta_i\omega_i)$$

$$\Phi^TK\Phi=\mathrm{diag}(\omega_i^2)$$

where the damping ratios $\{\zeta_i\}$ need not be equal or related.

Taking the simplest of these, an undamped m-input/m-output system, we have that $\Phi^TM\Phi=I$ and $\Phi^TK\Phi=\mathrm{diag}(\omega_i^2)$, where Φ is the modal matrix of the structure. Thus, $M=H^TH$ and $K=H^T\mathrm{diag}(\omega_i^2)H$, where $H=\Phi^{-1}$, so if H is partitioned as (H_1,H_2) with H_1 $(n\times m)$, etc., the ploynomial matrix $P_3(s)$ of Eq. (9) becomes

$$P_3(s)=s^2H_2^TH_2+H_2^T\,\mathrm{diag}(\omega_i^2)H_2 \tag{11}$$

Finding the structural zeros from this expression is complicated by the fact that H_2 is nonsquare. However, this can be overcome by means of the QR decomposition of H_2; this consists of a square upper triangular nonsingular R and a Q with orthonormal columns (i.e., $Q^TQ=I$) satisfying $H_2=QR$. Substituting this into Eq. (11) gives

$$P_3(s)=R^T[s^2I+Q^T\,\mathrm{diag}(\omega_i^2)Q]R \tag{12}$$

so the structural zeros are simply those s_i for which the bracketed term becomes singular. Therefore, they are given by $\{\pm jz_i;\ i=1,\ldots,n-m\}$, where $\{z_i^2\}$ are the eigenvalues of the positive semidefinite symmetric $Q^T\,\mathrm{diag}(\omega_i^2)Q$. But this matrix is of the form known[14] as an $(n-m)$- *section* of $\mathrm{diag}(\omega_i^2)$, so it is a classical result that its eigenvalues are related to those (trivially $\{\omega_i^2;\ i=1,\ldots,n\}$) of this diagonal matrix in the following manner:

$$\begin{gathered}\omega_1\leq z_1\leq\omega_{m+1}\\ \omega_2\leq z_2\leq\omega_{m+2}\\ \vdots\\ \omega_{n-m}\leq z_{n-m}\leq\omega_n\end{gathered} \tag{13}$$

This clearly reduces to to the standard single-input/single-output pole-zero interlacing for $m = 1$, while for $m > 1$ there is a "spreading" of the bounds on each individual zero, preventing such a simple relation. However, regardless of the number of inputs, it can be seen that the moduli of the transmission zeros of an undamped structure are bounded above and below by those of its poles; similar results hold trivially for their imaginary and (zero) real parts. Thus, relations of the form we are seeking do indeed apply in this case.

An interesting consequence of these zero bounds arises in the case of a spacecraft that is assumed, for design purposes, to behave as a rigid body. In this case, precisely one sensor/actuator pair will be used per rigid-body mode, so $\omega_1 = \cdots = \omega_m = 0$ and ω_{m+i} is the ith flexible mode natural frequency. Thus, the first zero lies below the first flexible mode here, the second zero below the second, etc., a very simple relation.

We now move on to the simplest form of damping, proportional damping $C = aM + bK$. It is easy to show that the moduli of the transmission zeros here are also bounded by those of the poles by relations of precisely the form given earlier. This can be proved by noting that the poles of a modally damped structure have moduli[7] $\{\omega_i\}$ that are independent of their damping ratios. Now, in general, if $\{M,C,K\}$ is modally damped, the subsystem $\{M_3,C_3,K_3\}$ in the zeros defining Eq. (9) is not itself guaranteed modal. However, it is modal for the special case of proportional damping; in fact, it is proportional, as clearly $C_3 = aM_3 + bK_3$. Thus, the moduli of the transmission zeros here are independent of their damping ratios, i.e., independent of a and b, so must be equal to the zeros moduli $\{z_i\}$ for the undamped case. Thus, Eq. (13) applies unchanged for a proportionally damped structure.

The preceding is by no means the only pole-zero relation that holds in this case. In fact, the following results will now be proved to be true for the more general class of all modally damped structures.

1) The real parts of all zeros are bounded above and below by those of the poles.
2) The moduli of all zeros are bounded above and below by those of the poles.
3) No zero has imaginary part smaller than the smallest imaginary part of any pole.
4) No zero has damping ratio greater than the greatest damping ratio of any pole.

The proof of each of these relations proceeds from the modal form of the zeros defining Eq. (9), which is [cf., Eq. (11)]

$$P_3(s) = H_2^T[s^2 I + s\ \mathrm{diag}(2\zeta_i\omega_i) + \mathrm{diag}(\omega_i^2)]H_2 \tag{14}$$

This becomes singular for $s = z$ a transmission zero, so there exists a nonzero $\boldsymbol{f}$ for which $P_3(z)\boldsymbol{f} = \boldsymbol{0}$: premultiplying by its conjugate transpose $\boldsymbol{f}^H$ and defining $\boldsymbol{g} = H_2\boldsymbol{f}$, this becomes

$$\boldsymbol{g}^H\ \mathrm{diag}(z^2 + 2\zeta_i\omega_i z + \omega_i^2)\boldsymbol{g} = 0 \tag{15}$$

which is simply the sum

$$\sum \gamma_i (z^2 + 2\zeta_i\omega_i z + \omega_i^2) = 0 \tag{16}$$

where $\gamma_i = |g_i|^2 \geq 0$. (Without loss of generality, we assume that $\boldsymbol{g}$ has been normalized to give $\Sigma\,\gamma_i = 1$). Now, remembering that the ith pole of this structure is $-\zeta_i\omega_i + j\omega_i\sqrt{[1-\zeta_i^2]}$ and writing the zero z as $x + jy$, it can be verified that the ith bracketed term in Eq. (16) is $(x^2 - y^2 + 2\zeta_i\omega_i x + \omega_i^2) + j2y(x + \zeta_i\omega_i)$. Both the real and imaginary parts of this sum over i must be zero for z any zero: considering the imaginary part first allows condition 1 to be proved, for it gives $2y\,\Sigma\,\gamma_i(x + \zeta_i\omega_i) = 0$. Therefore, either $y = 0$ (i.e., z is a purely real zero, which can be shown to be impossible for an under-damped structure as here) or

$$x = \sum \gamma_i(-\zeta_i\omega_i) \tag{17}$$

Thus, x is just the mean of the real parts $\{-\zeta_i\omega_i\}$ of the poles, weighted by the $\{\gamma_i\}$. But $\Sigma\,\gamma_i = 1$ and all $\gamma_i \geq 0$, therefore, x certainly lies somewhere between the maximum and minimum pole real parts, and condition 1 is established.

Relation 2 follows in a similar fashion from the real part of Eq. (16): this is $(x^2 - y^2)\,\Sigma\,\gamma_i + 2x\,\Sigma\,\gamma_i\zeta_i\omega_i + \Sigma\,\gamma_i\omega_i^2 = 0$, so replacing $\Sigma\,\gamma_i\zeta_i\omega_i$ by $-x$ and rearranging yields

$$x^2 + y^2 = \sum \gamma_i\omega_i^2 \tag{18}$$

Thus, $x^2 + y^2$, the square of the modulus of this zero, is a weighted mean of the squared pole moduli $\{\omega_i^2\}$, and condition 2 holds by analogous argument to condition 1.

The proof of conditions 3 and 4 requires use of the standard means inequality

$$\left(\sum \gamma_i c_i\right)^2 \leq \sum \gamma_i c_i^2 \tag{19}$$

for any $\{c_i\}$. Applying this to Eq. (17) gives $x^2 \leq \Sigma\,\gamma_i\zeta_i^2\omega_i^2$, which when substituted into Eq. (18) yields

$$y^2 = \sum \gamma_i\omega_i^2 - x^2 \geq \sum \gamma_i\omega_i^2(1 - \zeta_i^2)$$
$$\geq \left(\sum \gamma_i\omega_i\sqrt{[1-\zeta_i^2]}\right)^2 \tag{20}$$

by a second use of inequality (19). Thus, y is no smaller than the weighted mean of the imaginary parts of the poles, and so is certainly no smaller than the smallest of these, proving condition 3.

It only remains to prove condition 4. Let ζ_z be the damping ratio of the zero considered and ζ_* be the maximum damping ratio of any pole. Then we have $\zeta_z^2 = x^2/(x^2+y^2) = (\Sigma\,\gamma_i\zeta_i\omega_i)^2/\Sigma\,\gamma_i\omega_i^2$, by Eqs. (17) and (18), so clearly

$$\zeta_z^2 \leq \zeta_*^2\left(\sum \gamma_i\omega_i\right)^2 \Big/ \sum \gamma_i\omega_i^2 \leq \zeta_*^2 \tag{21}$$

by inequality (19). This proves the desired damping ratio bound.

Combining these four conditions allows a great deal to be said about the values that the transmission zeros of a structure may take, merely from a knowledge of its open-loop poles. This information is most readily expressed in graphical terms, as is shown for a general modally damped structure in Fig. 1. If all modes have equal damping ratios (as is often assumed to be the case), the permissible region for zeros becomes even simpler (see Fig. 2). The physical significance of these constraints is that an optimal regulator will never produce a closed-loop mode that decays at a slower rate (condition 1) than the slowest open-loop mode [although the closed-loop damping ratios may be lower than the open-loop values (condition 4)], or has lower natural (condition 2) or damped (condition 3) frequencies than the lowest in the open-loop system. Note, though, that a word of warning is needed for systems possessing rigid-body modes: these open-loop poles at the origin have the effect of making all lower bounds zero, therefore, in this case, it is possible for an optimal regulator to produce a closed-loop mode that oscillates or decays slower than any open-loop *flexible* mode. This point was illustrated by the simple "rigid-body control" spacecraft considered previously and will be returned to in the examples of the next section.

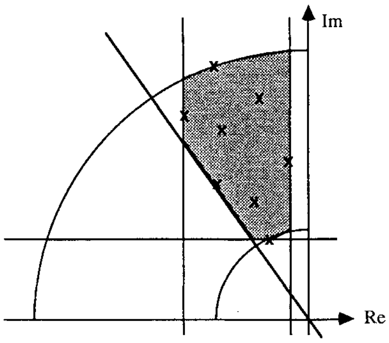

Fig. 1 Zero region for general modal damping.

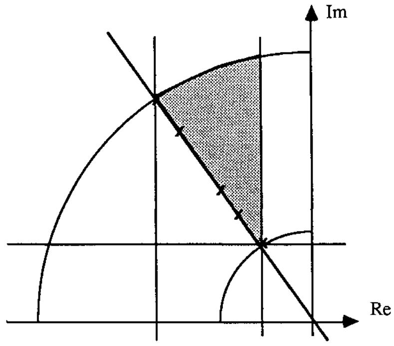

Fig. 2 Zero region for uniform modal damping.

The upper bounds in conditions 1 and 2 are of less physical interest than the corresponding lower bounds: this is fortunate, as the inaccuracy typical of the higher natural frequencies of LSS models makes them far less reliable quantities. They are also more affected when the order of the open-loop system model is increased, as is to be expected considering the extra high-frequency zeros this gives rise to. By contrast, the lower bounds in conditions 1–3 do not change at all for increasing n if the model considered is a true modal one, while the Rayleigh-Ritz convergence behavior of finite-element models[18,19] guarantees only small lower bound changes in such cases.

Finally, observe that these zero constraints were derived without reference to the specific locations chosen for sensor/actuator pairs. In fact, selecting these locations can be regarded as equivalent to selecting particular zero positions within the permissible region. Now, the effect of sensor/actuator positioning on the zeros is explicit [it defines the matrix $P_3(s)$ in Eqs. (8) and (9)] as is that of the zeros on closed-loop regulator performance (by the optimal root loci). The transmission zeros of a structure therefore provide a direct and novel way of quantifying the effects of sensor/actuator placement on the optimal closed-loop performance that results.

Examples

The zero bounds derived in the last section will now be illustrated by application to simple examples. The models considered are truncated true modal descriptions, serving to confirm the remarks following Eq. (8) that the zero analysis in such cases proceeds essentially as above.

"Rigid-Body Control" Spacecraft

Consider a three-axis spacecraft with sensor/actuator pairs chosen so that each one senses and excites precisely one rigid-body rotation mode. Suppose now that it is desired to calculate from first principles the single (as $q = n - m = 1$) transmission zero frequency, damping ratio, and mode shape for a model that includes the first flexible mode of this spacecraft as well.

Table 1 Transmission zero minima, maxima, and bounds

	Real	Imaginary	Modulus	Damping ratio
Lower bounds	0.0041	0.0826	0.0827	—
$\lambda = 0.8$	0.0257	0.5151	0.5157	0.0425
	0.1004	2.3016	2.3038	0.0498
$\lambda = 1.0$	0.0143	0.3634	0.3637	0.0392
	0.1172	2.5085	2.5112	0.0467
$\lambda = 0.3,0.8$	0.0382	0.8803	0.8811	0.0360
	0.0703	1.9495	1.9507	0.0434
$\lambda = 0.3,0.8,1.0$	0.0587	1.6093	1.6104	0.0365
	0.0587	1.6093	1.6104	0.0365
Upper bounds	0.1422	—	2.8433	0.0500

The relevant modal description is

$$\ddot{\boldsymbol{\eta}} + \text{diag}(0,0,0,2\zeta\omega)\dot{\boldsymbol{\eta}} + \text{diag}(0,0,0,\omega^2)\boldsymbol{\eta} = \boldsymbol{\Phi}^T\boldsymbol{u}, \quad \boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{\eta} \tag{22}$$

where $\boldsymbol{\eta}$ is the vector of modal amplitudes and, assuming unit moments of inertia for simplicity, the modal influence matrix $\boldsymbol{\Phi} = (I_3, \boldsymbol{v})$, where $\boldsymbol{v} = (\theta,\phi,\psi)^T$ consists of the sensors outputs resulting from the flexible vibration mode.

Now, we must have $\boldsymbol{y}(t)$ identically zero for forced vibration at the zero mode, so the special form of $\boldsymbol{\Phi}$ implies that $\boldsymbol{\eta}(t)$ must be proportional to $(\theta,\phi,\psi,-1)^T$ for all t for such vibration; thus, $\eta_1 = -\theta\eta_4$, $\dot{\eta}_1 = -\theta\dot{\eta}_4$, etc. But $\ddot{\eta}_1 = u_1$, so $u_1 = -\theta\ddot{\eta}_4$, etc., allowing $\boldsymbol{u}$ to be eliminated from the equation describing the dynamics of η_4 to give

$$(1 + \theta^2 + \phi^2 + \psi^2)\ddot{\eta}_4 + 2\zeta\omega\dot{\eta}_4 + \omega^2\eta_4 = 0 \tag{23}$$

This can be written as the damped harmonic motion equation $\ddot{\eta}_4 + 2\zeta_z\omega_z\dot{\eta}_4 + \omega_z^2\eta_4 = 0$, where

$$\omega_z = \omega/\sqrt{[1 + \theta^2 + \phi^2 + \psi^2]} \text{ and } \zeta_z = \zeta/\sqrt{[1 + \theta^2 + \phi^2 + \psi^2]} \tag{24}$$

so the transmission zero for this example has natural frequency ω_z and damping ratio ζ_z. These simple explicit results clearly imply that the general zero bounds (conditions 1–4) all hold here. Furthermore, the corresponding zero mode shape is seen to be of the form $\boldsymbol{\eta} \propto (\theta,\phi,\psi-1)^T$, while the damped harmonic control input required to excite this forced vibration is proportional to $(\theta,\phi,\psi)^T$.

Uniform Beam

Numerical results will now be derived for a four-mode model for transverse vibration of a uniform beam of length 25 m, width 0.1 m, and depth 0.01 m, and constructed of aluminum (density 2.7×10^3 kg/m^3, Young's modulus 7.0×10^{10} N/m^2). All modes are assumed to have a damping ratio of 5%—this artificially high value was chosen so as to allow reasonably large real parts for both poles and zeros.

Considering first the case where the beam is supported as a cantilever, the top and bottom lines of Table 1 list the transmission zero bounds given by applying conditions 1–4 to this structure. These bounds should hold regardless of the particular locations chosen for sensor/actuator pairs; to test this, the actual zeros resulting from various arrangements of colocated linear sensors and actuators were calculated. The remainder of Table 1 consists of the corresponding minimum and maximum values of transmission zero real and imaginary parts, moduli, and damping ratios; the dimensionless quantities λ given are the normalized distances from the clamped end of the beam of the one, two, or three sensor/actuator pairs considered.

Note that all zero bounds do indeed hold in every case, and that the number and placement of sensor/actuator pairs greatly influence the specific transmission zeros that result, and so the optimal closed-loop performance attainable with each configuration. As an approximate general rule, the greater the number of sensors m, the faster the zero modes; this follows from the fact[7] that increasing the number of constraints increases the frequencies of the resulting constrained modes, and is reflected, for the undamped case, in the role of m in inequality (13).

Finally, an interesting comparison with the preceding is given by considering the same beam, but now in the free-free end condition. This configuration has a rigid-body mode so, as previously noted, it is possible to obtain a transmission zero that is slower than any open-loop bending mode. This turns out to indeed be the case if a single colocated angular sensor/actuator pair is placed at one tip; the slowest zero mode then has natural frequency 0.1520 rad/s, considerably below the lowest bending frequency of 0.5262 rad/s. This observation, together with the fact that this zero mode is very lightly damped (1.12%), shows that closed-loop regulated performance will be quite poor for this sensor/actuator arrangement. While this merely confirms what could be deduced from physical intuition for this very simple example, it clearly suggests the likely value of transmission zeros for selecting sensor/actuator locations on more complicated, and realistic, structures.

Conclusions

This paper has emphasized the fact that the transmission zeros of a system are just as important as its poles for a full description of its dynamics, and, in particular, of the closed-loop speed of response obtainable by applying an optimal regulator to it. If the system under consideration is a large space structure with colocated sensors and actuators, these zeros were shown to lie in a region of the complex plane that is defined by its natural frequencies and damping ratios; for an undamped single-input/single-output structure, this reduces to the classical pole-zero interlacing property. Selecting particular sensor/actuator locations can be regarded as specifying where in the permissible region transmission zeros actually lie, so quantifying the effect of sensor/actuator placement on closed-loop system performance. These points were illustrated by simple examples.

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